Test 1 / Numerical Mathematics 1 / May 11th 2020, University of Groningen

No calculator and additional material is allowed.
All answers need to be justified using mathematical arguments.
Total time: 1 hour 30 minutes (time includes upload of the PDF with your answers to Nestor) + 10 minutes (if special needs)
Remember: oral "checks" may be run afterwards.

Grade = (obtained points) + 1

Consider a function $g \in C^5([a, b])$, and the approximating polynomial $z(x)$ defined as:

$$z(x) = g(a) + g'(a)(x - a) + \frac{1}{2}g''(a)(x - a)^2$$

(a) $\boxed{2}$ Compute an error bound ϵ (independent on x) such that $|g(x) - z(x)| < \epsilon$, $\forall x \in [a, b]$, where a, b and g appear explicitly in ϵ. Verify that ϵ goes to zero when $b \to a$.

(b) $\boxed{2}$ Use the approximating polynomial $z(x)$ to define a numerical integration rule of $g(x)$ over $[a, b]$.

(c) $\boxed{1.5}$ What is the degree of exactness of the resulting integration method? Justify rigorously your answer.

(d) $\boxed{1.0}$ Draw how $z(x)$ approximates the function $\cos x$ over $x \in [-\pi, \pi]$. Illustrate (shading some region of the plot, for instance) the numerical integration rule.

(e) $\boxed{2.5}$ Assume perturbed measurements of g of the form $\hat{g}^{(i)}(a) = g^{(i)}(a) + \delta_i/(b - a)^i$, with the superindex (i) denoting the ith-derivative. The approximating function now has the form:

$$\hat{z}(x) = \hat{g}(a) + \hat{g}'(a)(x - a) + \frac{1}{2}\hat{g}''(a)(x - a)^2$$

Prove that:

$$|g(x) - \hat{z}(x)| \leq \epsilon + \max_i |\delta_i| \sum_{i=0}^{2} \frac{1}{i!}$$